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A REMARK ON STRONGLY EXPOSING FUNCTIONALS

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ABSTRACT. By using the concept of farthest points, we show that the set of strongly exposing functionals of a weakly compact convex subset in a Banach space X is a dense G_δ in X^*. The construction also gives a new proof of existence of strongly exposed points in weakly compact convex sets.

Let K be a convex subset in a Banach space X, a point $x \in K$ is called a *strongly exposed point* of K if there exists an $f \in X^*$ such that (i) $f(x) > f(y)$ for all $y \neq x$ in K, (ii) for any sequence (x_n) in K with $f(x_n) \to f(x)$, $x_n \to x$ in norm. We call the above f a *strongly exposing functional* of K and use K^Λ to denote the set of strongly exposing functionals of K. Lindenstrauss [5] and Troyanski [6] proved that if K is a weakly compact convex subset in X, then K is the closed convex hull of its strongly exposed points. In [1], Anantharaman showed that if K is the closed convex hull of the range of a vector-valued measure (hence K is weakly compact) then K^Λ is a dense G_δ in X^*. A similar conclusion has also been obtained by the author for weakly compact convex subsets in certain classes of Banach spaces [4]. In this note, by modifying the method in [4], we prove

THEOREM 1. *Let K be a weakly compact convex subset in a Banach space X; then K^Λ is a dense G_δ in X^*.*

In the proof, we will need the following propositions.

PROPOSITION 2 (TROYANSKI). *Let X be a weakly compact generated Banach space; then X admits an equivalent locally uniformly convex norm.*

PROPOSITION 3 (LAU). *Let K be a weakly compact subset in a Banach space X; then the set*

$$\{x \in X: \|x - z\| = \sup\{\|x - y\|: y \in K\} \text{ for some } z \in K\}$$

is a dense G_δ in X.

We call the point z in the above proposition a *farthest point* of K [2], [3]. It is known that if X is locally uniformly convex, then a farthest point of a

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bounded convex subset is also a strongly exposed point.

PROOF OF THE THEOREM. Note that

$$K^{\Lambda} = \bigcap_{n=1}^{\infty}\left\{f \in X^*: \operatorname{diam}\left\{x \in K: f(x) > \sup_{y\in K} f(y) - a\right\} < \frac{1}{n} \text{ for some } a > 0\right\}$$

and the set on the right side is a G_δ [1], [4]; hence it suffices to show the density of K^{Λ} in X^*. By a remark in [4] and Proposition 2, we may assume that X is weakly compact generated (say, by K) and locally uniformly convex. Let $f \in X^*$ with $\|f\| = 1$. For $\varepsilon^{>0}$, let $C = f^{-1}(0) \cap 2\varepsilon^{-1}B$ where B is the closed unit ball of X. By a homothetic translation, we may let $K \subseteq B$ but $K \nsubseteq C$ (note that K^{Λ} is unchanged). We will construct a point $z \in K$ which is a strongly exposed point of the closed set conv $(K \cup C)$. The corresponding strongly exposing functional g of conv $(K \cup C)$ with $\|g\| = 1$ will satisfy $\|f - g\| \leqslant \varepsilon$ and also strongly exposes K at z (for details, cf. [4, Theorem 2.4]); hence this completes the proof.

Choose a point $x_1 \in K\backslash C$ such that the set

$$S = \{\alpha x_1 + \beta y: |\alpha|^2 + |\beta|^2 \leqslant 1, y \in C\}$$

does not contain K (we neglect the case that K is a singleton, x_1 may be chosen as midpoint of some line segment of K not lying in C). Since C is an absorbent subset of the hyperplane $f^{-1}(0)$, S is an absorbent subset of X. Let $|||\cdot|||$ be the norm defined by S; then $|||\cdot|||$ is locally uniformly convex and equivalent to the original norm. There exists $x_2 \in K\backslash S$ with $|||x_2||| - 1 = \alpha > 0$. By Proposition 3, there exist points $w \in X$, $z \in K$ with $|||w||| \leqslant \alpha/2$ and $\beta = |||w - z||| = \sup\{|||w - y|||: y \in K\}$. For any point $y \in C$,

$$|||y - w||| \leqslant |||y||| + \alpha/2 \leqslant 1 + \alpha/2 < \beta.$$

Hence z is also a farthest point of conv $(K \cup C)$. It follows that z is a strongly exposed point of conv $(K \cup C)$. Q.E.D.

We remark that the above construction yields another proof of the existence of strongly exposed points in weakly compact sets as in [5]. Moreover, we have

COROLLARY 4. *Let K be a weakly compact convex subset in a Banach space X; then for any bounded closed convex subset C such that $K \nsubseteq C$, there exists a point $x \in K$ which strongly exposes* conv $(K \cup C)$.

PROOF. It follows easily from the above theorem and Theorem 2.4 in [4]: if K is a bounded closed convex subset in X, then K^{Λ} is a dense G_δ if and only if for any bounded closed convex subset C such that $K \nsubseteq C$, there exists a point $x \in K$ which is a strongly exposed point of $\overline{\text{conv}}\,(K \cup C)$.

References

1. R. Anantharaman, *On exposed points of the range of a vector measure.* II, Proc. Amer. Math. Soc. **55** (1976), 334–338.

2. M. Edelstein and J. E. Lewis, *On exposed and farthest points in normed linear spaces*, J. Austral. Math. Soc. **12** (1971), 301–308. MR **46** #9694.

3. K. Lau, *Farthest points in weakly compact sets*, Israel J. Math. **22** (1975), 168–174.

4. ———, *On strongly exposing functionals*, J. Austral. Math. Soc. (to appear).

5. J. Lindenstrauss, *On operators which attain their norm*, Israel J. Math. **1** (1963), 139–148. MR **28** #3308.

6. S. L. Troyanski, *On locally uniformly convex and differentiable norms in certain non-separable Banach spaces*, Studia Math. **37** (1970/71), 173–180. MR **46** #5995.

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